Leader-following Consensus Problems with a Time-varying Leader under Measurement Noises

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Abstract

In this paper, we consider a leader-following consensus problem for networks of continuous-time integrator agents with a time-varying leader under measurement noises. We propose a neighbor-based state-estimation protocol for every agent to track the leader, and time-varying consensus gains are introduced to attenuate the noises. By combining the tools of stochastic analysis and algebraic graph theory, we study mean square convergence of this multi-agent system under directed fixed as well as switching interconnection topologies. Sufficient conditions are given for mean square consensus in both cases. Finally, a numerical example is given to illustrate our theoretical results.

Keywords: consensus problems; multi-agent system; leader-following; stochastic system.

1. Introduction

In recent years, there has been an increasing research interest in the distributed coordination for multi-agent systems. This is partly due to its broad applications in many areas such as cooperative control of unmanned aerial vehicles, formation control [1, 26, 27] and swarming behaviors of social living beings [11, 16, 30].

Consensus problems have a long history in computer science and formed the foundation of the field of distributed computing [12]. In consensus control, it is critical to design a decentralized network algorithm based on neighborhood information for agents to reach an agreement on their states, asymptotically in some sense. For a variety of consensus algorithms and convergence results we refer the reader to the comprehensive surveys [17, 24] and references therein. Most researches in the previous literature assume the exchange of messages between agents is error-free. However, this is only an ideal approximation for real communication processes. Recently, consensus of dynamic networks with random measurement noises has attracted the attention of some researchers. In [8, 25], the authors introduce time-varying consensus gains and design control schemes based on a Kalman filter structure. The decreasing consensus gain $a(k)$ (where k is the discrete time instant) in the protocols is proposed in [5] to attenuate the measurement noises in a strongly connected circulant network. The analysis in [5] is generalized to strongly connected digraphs in [7] and digraphs containing a spanning tree in [6] by the same authors. The work in [10] deals with discrete-time average consensus problems in switching balanced digraphs under stochastic communication noises, while [9] investigates the continuous-time average consensus control with fixed topology and Gaussian communication noises. The authors in [13] treat a continuous-time leader-following consensus control under measurement noises with a constant state leader.

In this paper, motivated by the above works, we consider a leader-following consensus problem for networks of continuous-time integrator agents with a time-varying leader in directed fixed and switching topologies. The control input of each agent is based on the

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measurement of its neighbors' states and some estimated data of the leader which are both corrupted by white noises. We design a leader-following consensus protocol such that the leader has an underlying dynamics and some variables (e.g. velocity and acceleration) of the leader cannot be measured and every follower can obtain the measured information (e.g. position) of the leader only when they are connected with the leader directly. The collective behavior of self-organized groups of agents with active (or dynamical) leaders is one of the most interesting topics in distributed cooperative control. However, as [22] suggests, the extension of consensus algorithms from a constant reference to a time-varying one is non-trivial. Some related results can be found e.g. in [3, 20, 31], where the systems considered are all deterministic and free of noise.

Inspired by [5, 9, 13], we introduce time-varying consensus gains in the followers control protocol to attenuate the measurement noises, which lead to a time-varying stochastic differential equation of the system. The state matrix of the equation is time-dependent and no longer a Laplacian matrix, and is neither symmetric nor diagonalizable. To implement the convergence study, we merge stochastic analysis and algebraic graph theory, by developing a Lyapunov-based approach and addressing the Itô integral by the stopping time truncation method. Firstly, we derive a sufficient condition for the state of each follower to converge to that of the leader in mean square under fixed and directed interconnection topology. Then it is shown that the algorithm also render each follower track the leader in mean square under switching topology when the subgraph induced by the followers is balanced.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and present the leader-following consensus protocol. Section 3 contains the convergence analyses under directed fixed and switching interaction topologies. A numerical example is given in Section 4 and we conclude the paper in Section 5.

2. Problem formulation

Before we proceed, some basic concepts on graph theory (see e.g. [2]) are provided as below.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with the set of vertices $\mathcal{V} = \{1, 2, \cdots, n\}$ and the set of arcs $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The vertex i in $\mathcal{G}$ represents the ith agent, and a directed edge $(i, j) \in \mathcal{E}$ means that agent j can directly receive information from agent i. The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} |\ (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements and $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. The in-degree and out-degree of vertex i are defined as $d_{in}(i) = \sum_{j=1}^{n} a_{ij}$ and $d_{out}(i) = \sum_{j=1}^{n} a_{ji}$, respectively. If $d_{in}(i) = d_{out}(i)$ for $i = 1, 2, \cdots, n$, then the digraph $\mathcal{G}$ is called balanced [18]. The Laplacian of $\mathcal{G}$ is defined as $L = D - \mathcal{A}$, where $D = \mathrm{diag}(d_{in}(1), d_{in}(2), \cdots, d_{in}(n))$. A digraph $\mathcal{G}$ is called strongly connected if there is a directed path from i to j between any two distinct vertices $i, j \in \mathcal{V}$. There exists a directed path from vertex i to vertex j, then j is said to be reachable from i. For every vertex in digraph $\mathcal{G}$, if there is a path from vertex i to it, then we say i is globally reachable in $\mathcal{G}$. This is much weaker than strong connectedness.

Here, we consider a system consisting of $n+1$ agents, in which an agent indexed by 0 is assigned as the leader and the other agents indexed by $1, 2, \cdots, n$ are referred as follower agents. The information interaction topology among n followers are described by the digraph $\mathcal{G}$ as defined above; and the whole system including $n + 1$ agents is conveniently

modeled by a weighted digraph $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}}, \overline{\mathcal{A}})$ with $\overline{\mathcal{V}} = \{0, 1, \cdots, n\}$ and

$$\overline{\mathcal{A}} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

where the lower right block submatrix of order n can be viewed as $\mathcal{A}$. We define a diagonal matrix $B = \mathrm{diag}(b_1, b_2, \cdots, b_n)$ to be the leader adjacency matrix associated with $\overline{\mathcal{G}}$, where $b_i = a_{i0} \geq 0$ and $b_i > 0$ if and only if $0 \in \mathcal{N}_i(\overline{\mathcal{G}})$. Here, $\mathcal{N}_i(\overline{\mathcal{G}})$ is the set of neighbors of agent i in $\overline{\mathcal{G}}$.

The continuous-time dynamics of n followers is described as follows:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \cdots, n, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the ith agent, and $u_i(t) \in \mathbb{R}$ is the control input. The leader of this considered multi-agent system is described by a double integrator of the form:

$$\begin{cases} \dot{x}_0(t) = g(t)v_0(t) \\ \dot{v}_0(t) = a_0(t) \\ y(t) = x_0(t) \end{cases} \tag{2}$$

where $g(t) : [0, \infty) \to (0, \infty)$ is piecewise continuous, $y(t)$ is the measured output and $a_0(t)$ is the input. We assume $g(t)$ and $a_0(t)$ are known, that is, the dynamical behavior of the leader is precisely known (c.f. Remark 1). On the other hand, $y(t) = x_0(t)$ is the only data that may be gotten by the followers when they are connected to the leader directly. Since $v_0(t)$ cannot be measured, we have to estimate $v_0(t)$ in a distributed way during the evolution. The estimate of $v_0(t)$ by agent i is denoted by $v_i(t)$, $i = 1, 2, \cdots, n$.

In our model, the ith agent receives information from its neighbors with measurement noises:

$$y_{ji}(t) = x_j(t) + \sigma_{ji} n_{ji}(t), \quad j \in \mathcal{N}_i, \tag{3}$$

$$y_{0i}(t) = x_0(t) + \sigma_{0i} n_{0i}(t), \tag{4}$$

where $y_{ji}(t)$ $(i \in \mathcal{V}, j \in \overline{\mathcal{V}})$ denotes the measurement of the jth agent's state $x_j(t)$ by the ith agent. The $\{n_{ji}(t) |\ j \in \overline{\mathcal{V}}, i \in \mathcal{V}\}$ are independent standard white noises and $\sigma_{ji} \geq 0$ is the noise intensity.

A group of controls $\mathcal{U} = \{u_i |\ i = 1, 2, \cdots, n\}$ is called a measurement-based distributed protocol [9], if $u_i(t) \in \sigma\big(x_i(s), \bigcup_{j\in\mathcal{N}_i} y_{ji}(s),\ 0 \leq s \leq t\big)$, for $t \geq 0$, $i = 1, 2, \cdots, n$. Herein $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ-algebra generated by a family of random variables $\{\xi_\lambda, \lambda \in \Lambda\}$. The so-called leader-following consensus problem is to design a measurement-based distributed protocol such that each follower's state will converge to the leader's in some sense as time goes on.

Consequently, we propose the distributed control protocol which consists of two parts:

- a neighbor-based feedback law:

$$u_i(t) = h(t)\Big(\sum_{j\in\mathcal{N}_i} a_{ij}(y_{ji}(t) - x_i(t)) + b_i(y_{0i}(t) - x_i(t))\Big) + g(t)v_i(t), \quad i = 1, 2, \cdots, n \tag{5}$$

where $t \geq 0$ and $h(t) : [0, \infty) \to (0, \infty)$ is a piecewise continuous function, called a time-varying consensus gain [9].

• a dynamic neighbor-based system to estimate $v_0(t)$:

$$\dot{v}_i(t) = a_0(t) + \gamma h(t)\Big(\sum_{j\in\mathcal{N}_i} a_{ij}(y'_{ji}(t) - x_i(t)) + b_i(y'_{0i}(t) - x_i(t))\Big), \quad i = 1, 2, \cdots, n \quad (6)$$

where $0 < \gamma < 1$ is some constant, and moreover $y'_{ji}(t)$ and $y'_{0i}(t)$ are independent copies of $y_{ji}(t)$ and $y_{0i}(t)$, respectively. In other words, we have

$$y'_{ji}(t) = x_j(t) + \sigma_{ji}n'_{ji}(t), \quad j \in \mathcal{N}_i, \quad (7)$$

$$y'_{0i}(t) = x_0(t) + \sigma_{0i}n'_{0i}(t), \quad (8)$$

where $\{n'_{ji}(t) |\ j \in \overline{\mathcal{V}}, i \in \mathcal{V}\}$ are independent standard white noises and independent with $\{n_{ji}(t) |\ j \in \overline{\mathcal{V}}, i \in \mathcal{V}\}$.

The set of neighbors $\mathcal{N}_i$ of agent i in (5) and (6) varies when the interconnection topology is switching and we defer the corresponding protocol formulation to Section 3.2.

Remark 1. *We take individual state x_i as scalar for simplicity in (1) and it can be extended to multi-dimensional scenarios as studied in [20, 31] without much effort. For example, if $x_i \in \mathbb{R}^2$, it can be thought as the position of agent i moving in a plane. Therefore, gv_0 and $ga_0 + \dot{g}v_0$ are the velocity and acceleration of the leader respectively, which are known since the exact dynamics of the leader is assumed.*

Remark 2. *We separate a factor g from the 'velocity term' of the leader in (2) in order to tone the decreasing consensus gain h, which appears to be a notable feature distinct from some kinds of uncertain environment (see e.g. [28, 29]), where a random term is directly appended behind the equation of the system. In such works, the consensus gains are supposed to have positive lower bound.*

Remark 3. *From (5) and (6) it is clear that the designed protocol for the ith agent is indeed a measurement-based distributed protocol since it relies only on the state of itself and its neighbors.*

Let $x(t) = (x_1(t), \cdots, x_n(t))^T$ and $v(t) = (v_1(t), \cdots, v_n(t))^T$. Denote the ith row of the matrix $\mathcal{A}$ by α_i, and $\Sigma_i := \mathrm{diag}(\sigma_{1i}, \cdots, \sigma_{ni})$ for $i = 1, 2, \cdots, n$. Hence $\Sigma := \mathrm{diag}(\alpha_1\Sigma_1, \cdots, \alpha_n\Sigma_n)$ is an $n \times n^2$ dimensional block diagonal matrix. Let $n_0(t) = (n_{01}(t), \cdots, n_{0n}(t))^T$ and $n_i(t) = (n_{1i}(t), \cdots, n_{ni}(t))^T$ for $i = 1, 2, \cdots, n$. In addition, $n'_0(t)$ and $n'_i(t)$ can be defined in a similar way. The juxtaposed matrix $Q := (B, \Sigma)$ is an $n \times n(n+1)$ dimensional block matrix. Combining (1) with (5) and (6), we may write the protocol in a matrix form:

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = & -h(t)(L + B)x(t) + h(t)B1x_0(t) + g(t)v(t) + h(t)QZ(t) \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} = & a_0(t)1 - \gamma h(t)(L + B)x(t) + \gamma h(t)B1x_0(t) + \gamma h(t)QZ'(t) \end{cases} \quad (9)$$

where $Z(t) = (n_0^T(t), n_1^T(t), \cdots, n_n^T(t))^T$ and $Z'(t) = ({n'_0}^T(t), {n'_1}^T(t), \cdots, {n'_n}^T(t))^T$ are two $n(n+1)$ dimensional independent standard white noise sequences, and $1 = (1, \cdots, 1)^T \in \mathbb{R}^n$. The system (9) may be further written in the form of the Itô stochastic differential equations:

$$\begin{cases} \mathrm{d}x(t) = & -h(t)(L + B)x(t)\mathrm{d}t + h(t)B1x_0(t)\mathrm{d}t + g(t)v(t)\mathrm{d}t + h(t)G\mathrm{d}W_1(t) \\ \mathrm{d}v(t) = & a_0(t)1\mathrm{d}t - \gamma h(t)(L + B)x(t)\mathrm{d}t + \gamma h(t)B1x_0(t)\mathrm{d}t + \gamma h(t)G\mathrm{d}W_2(t) \end{cases} \quad (10)$$

where $W_1(t) = (W_{11}(t), \cdots, W_{1n}(t))^T$ and $W_2(t) = (W_{21}(t), \cdots, W_{2n}(t))^T$ are two n dimensional standard Brownian motions which are independent with each other, and $G := \mathrm{diag}\Big(\sqrt{b_1^2 + \sum_{j\in\mathcal{N}_1} \sigma_{j1}^2 a_{1j}^2}, \cdots, \sqrt{b_n^2 + \sum_{j\in\mathcal{N}_n} \sigma_{jn}^2 a_{nj}^2}\Big)$.

3. Convergence analysis

In this section we will give the convergence analysis of the system (10) and show that the state of every follower will track that of the leader in the sense of mean square convergence, that is, $E\|x(t) - x_0(t)1\| \to 0$, as $t \to \infty$. Here $\|\cdot\|$ denotes Frobenius norm. Two different cases, fixed topology and switching topology, are explored.

Remark 4. *Mean square consensus protocols for stochastic systems are first introduced in [5] and then further studied by several researchers (e.g. [6, 7, 9, 10, 13]). Mean square convergence seems to be an important alternative for almost sure convergence in consensus problems under noisy environments.*

For a given symmetric matrix A, let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalue, respectively. To get the main result, we need the following assumptions:

(A1) The vertex 0 is globally reachable in $\overline{\mathcal{G}}$.

(A2) There is a $\delta > 0$, such that $\frac{h(t)}{g(t)} > \frac{\lambda_{\max}(P)}{2\gamma(1-\gamma^2)} + \delta$ for $t \geq 0$.

Here, P is a positive definite matrix defined by Equation (13), see below.

(A3) $\int_0^\infty h(s)\mathrm{d}s = \infty$.

(A4) $\int_0^\infty h^2(s)\mathrm{d}s < \infty$.

Remark 5. *Assumption (A1) is imposed on the network topology, which is much weaker than strong connectedness. The technical Assumption (A2) roughly means that g is comparable with the consensus gain h. Assumptions (A3) and (A4) are called convergence condition and robustness condition respectively in [9], and which are standard assumptions often used in the stochastic approximation [15].*

3.1. Fixed topology

Let $x^* = x - x_0 1$ and $v^* = v - v_0 1$. We then obtain an error dynamics of (10) as follows:

$$\mathrm{d}\varepsilon(t) = F(t)\varepsilon(t)\mathrm{d}t + G(t)\mathrm{d}W(t), \quad t \geq 0 \tag{11}$$

where $\varepsilon(t) = (x^*(t), v^*(t))^T$, $W(t) = (W_1(t), W_2(t))^T$ and

$$F(t) = \begin{pmatrix} -h(t)(L+B) & g(t)I_n \\ -\gamma h(t)(L+B) & 0 \end{pmatrix}, \qquad G(t) = h(t)\begin{pmatrix} G \\ \gamma G \end{pmatrix} := h(t)\widetilde{G}.$$

Here I_n denotes the $n \times n$ dimensional identity matrix.

We will need a lemma for Laplacian matrix.

Lemma 1.[2, 23] *The Laplacian matrix L of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, L has exactly one zero eigenvalue if and only if there is a globally reachable vertex in $\mathcal{G}$.*

The main result in this section is given as follows:

Theorem 1. *For system (1) with the consensus protocols (5) and (6), if Assumptions (A1)-(A4) hold, then*

$$\lim_{t\to\infty} E\|\varepsilon(t)\|^2 = 0. \tag{12}$$

Proof. By Assumption (A1) and Lemma 1, we know $L + B$ is a positive stable matrix, or in other words, $-L - B$ is a stable matrix. From Lyapunov theorem, there is a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$(L+B)^T P + P(L+B) = I_n. \tag{13}$$

Let $\widetilde{P} = \begin{pmatrix} P & -\gamma P \\ -\gamma P & P \end{pmatrix}$ and define a Lyapunov function $V(t) = \varepsilon^T(t)\widetilde{P}\varepsilon(t)$. Since $0 < \gamma < 1$, $\widetilde{P}$ is a positive definite matrix. In fact, we have $\lambda_{\min}(\widetilde{P}) = (1-\gamma)\lambda_{\min}(P)$ and $\lambda_{\max}(\widetilde{P}) = (1+\gamma)\lambda_{\max}(P)$. Utilizing Itô formula and (11), we have

$$\mathrm{d}V(t) = \varepsilon^T(t)\big(\widetilde{P}F(t) + F^T(t)\widetilde{P}\big)\varepsilon(t)\mathrm{d}t + \mathrm{tr}(G^T(t)\widetilde{P}G(t))\mathrm{d}t + 2\varepsilon^T(t)\widetilde{P}G(t)\mathrm{d}W(t).$$

Here $\mathrm{tr}(\cdot)$ means the trace of a matrix. From the Lyapunov equation (13), we get

$$\widetilde{P}F(t) + F^T(t)\widetilde{P} = -h(t)\begin{pmatrix} (1-\gamma^2)I_n & -P\frac{g(t)}{h(t)} \\ -P\frac{g(t)}{h(t)} & 2\gamma P\frac{g(t)}{h(t)} \end{pmatrix} := -h(t)\widetilde{Q}(t).$$

Invoking the Haynsworth inertia additivity formula [21], Assumption (A2) and the positive definiteness of P, we know that $\widetilde{Q}(t)$ is positive definite with $\rho := \min_{t\geq 0}\big\{\lambda_{\min}(\widetilde{Q}(t))\big\} > 0$.

Thereby, we have

$$\mathrm{d}V(t) \leq -h(t)\frac{\rho}{\lambda_{\max}(\widetilde{P})}V(t)\mathrm{d}t + h^2(t)\mathrm{tr}(\widetilde{G}^T\widetilde{P}\widetilde{G})\mathrm{d}t + 2h(t)\varepsilon^T(t)\widetilde{P}\widetilde{G}\mathrm{d}W(t). \tag{14}$$

Next we want to prove

$$E\int_{t_0}^{t} h(s)\varepsilon^T(s)\widetilde{P}\widetilde{G}\mathrm{d}W(s) = 0, \quad \forall\ 0 \leq t_0 \leq t. \tag{15}$$

For any given $T \geq t_0 \geq 0$ and $K \in \mathbb{N}$, let $\tau_K^{t_0,T} = T \wedge \inf\{t \geq t_0 |\ \varepsilon^T(t)\widetilde{P}\varepsilon(t) \geq K\}$, which is a stopping time. By (14) we have, for $t_0 \leq t \leq T$,

$$\begin{aligned}
&E\big(V(t\wedge\tau_K^{t_0,T})1_{[t\leq\tau_K^{t_0,T}]}\big) - EV(t_0) \\
&\leq -\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_{t_0}^{t} h(s)E\big(V(s\wedge\tau_K^{t_0,T})1_{[s\leq\tau_K^{t_0,T}]}\big)\mathrm{d}s + \mathrm{tr}(\widetilde{G}^T\widetilde{P}\widetilde{G})\int_{t_0}^{t} h^2(s)\mathrm{d}s \\
&\leq \mathrm{tr}(\widetilde{G}^T\widetilde{P}\widetilde{G})\int_{t_0}^{T} h^2(s)\mathrm{d}s.
\end{aligned}$$

This implies that there is a constant $C_{t_0,T} > 0$ such that

$$E\big(V(t\wedge\tau_K^{t_0,T})1_{[t\leq\tau_K^{t_0,T}]}\big) \leq C_{t_0,T}, \quad \forall\ t_0 \leq t \leq T.$$

Since $\lim_{K\to\infty} t\wedge\tau_K^{t_0,T} = t$ a.s., for $t_0 \leq t \leq T$, by Fatou lemma, we derive

$$\sup_{t_0\leq t\leq T} EV(t) \leq C_{t_0,T}.$$

Accordingly,

$$E\int_{t_0}^{t} h^2(s)V(s)\mathrm{d}s \leq \sup_{t_0\leq s\leq t} EV(s)\cdot\int_0^T h^2(s)\mathrm{d}s < \infty, \quad \forall\ 0 \leq t_0 \leq t.$$

Combining this with

$$E\int_{t_0}^{t} h^2(s)\|\varepsilon^T(s)\widetilde{P}\widetilde{G}\|^2\mathrm{d}s \leq \|\widetilde{P}\|\|\widetilde{G}\|^2 E\int_{t_0}^{t} h^2(s)V(s)\mathrm{d}s$$

yields (15) (c.f. [19]).

Now employing (14) and (15), we obtain

$$EV(t) - EV(0) \leq -\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_0^t h(s)EV(s)\mathrm{d}s + \mathrm{tr}(\widetilde{G}^T\widetilde{P}\widetilde{G})\int_0^t h^2(s)\mathrm{d}s, \quad \forall\, t \geq 0.$$

Thus, from the comparison principle [14],

$$\begin{aligned} EV(t) \ \leq \ & EV(0)\exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_0^t h(s)\mathrm{d}s\Big) \\ & + \mathrm{tr}(\widetilde{G}^T\widetilde{P}\widetilde{G})\int_0^t h^2(s)\exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_s^t h(u)\mathrm{d}u\Big)\mathrm{d}s. \end{aligned} \tag{16}$$

Clearly, by Assumption (A3) the first term on the right hand side of (16) tends to 0, as $t \to \infty$. For any $\eta > 0$, by Assumption (A4), there exists some $s_0 > 0$ such that $\int_{s_0}^{\infty} h^2(s)\mathrm{d}s < \eta$. Hence,

$$\begin{aligned} & \int_0^t h^2(s)\exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_s^t h(u)\mathrm{d}u\Big)\mathrm{d}s \\ & = \int_0^{s_0} h^2(s)\exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_s^t h(u)\mathrm{d}u\Big)\mathrm{d}s + \int_{s_0}^t h^2(s)\exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_s^t h(u)\mathrm{d}u\Big)\mathrm{d}s \\ & \leq \exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_{s_0}^t h(u)\mathrm{d}u\Big)\int_0^{s_0} h^2(s)\mathrm{d}s + \int_{s_0}^t h^2(s)\mathrm{d}s \\ & \leq \exp\Big(-\frac{\rho}{\lambda_{\max}(\widetilde{P})}\int_{s_0}^t h(u)\mathrm{d}u\Big)\int_0^{\infty} h^2(s)\mathrm{d}s + \eta, \quad \forall\, t \geq s_0 \end{aligned}$$

By Assumptions (A3), (A4) and the arbitrariness of η, the last expression tends to zero, as $t \to \infty$. Therefore, (16) yields $\lim_{t\to\infty} EV(t) = 0$. Note that

$$\|\varepsilon(t)\|^2 \leq \frac{V(t)}{\lambda_{\min}(\widetilde{P})}$$

which concludes the proof. □

Remark 6. *As is known, the solution to Lyapunov matrix equation may be obtained by using Kronecker product. Thus the positive definite matrix P involved in Assumption (A2) can be given explicitly.*

Remark 7. *Theorem 1 implies that in the fixed topology, under Assumptions (A1)-(A4), the designed protocol guarantees the state of each follower tracks that of the leader in mean square.*

3.2. Switching topology

In this section we deal with the convergence of the protocol under switching topology.

Let $\sigma(t) : [0, \infty) \to \mathcal{S}_{\mathcal{H}} = \{1, 2, \cdots, N\}$ be a switching signal that determines the communication topology. The set $\mathcal{H}$ is a set of digraphs with the common vertex set $\overline{\mathcal{V}}$ and can be denoted as $\mathcal{H} = \{\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2, \cdots, \overline{\mathcal{G}}_N\}$, where N is the total number of digraphs in $\mathcal{H}$. Naturally, let $\mathcal{G}_{\sigma(t)}$ be the subgraph of $\overline{\mathcal{G}}_{\sigma(t)}$ induced by $\mathcal{V}$. Thereby, we rewrite the

consensus protocols (5) and (6) as:

$$u_i(t) = h(t)\Big(\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(y_{ji}(t)-x_i(t)) + b_i(\sigma(t))(y_{0i}(t)-x_i(t))\Big) + g(t)v_i(t), \quad i=1,2,\cdots,n \quad (17)$$

and

$$\dot{v}_i(t) = a_0(t) + \gamma h(t) \cdot \Big(\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(y'_{ji}(t)-x_i(t)) + b_i(\sigma(t))(y'_{0i}(t)-x_i(t))\Big), \quad i=1,2,\cdots,n \quad (18)$$

where, $\mathcal{N}_i(\sigma(t))$ is the set of neighbors of agent i in the digraph $\mathcal{G}_{\sigma(t)}$ formed by n followers; $a_{ij}(\sigma(t))$ is the (i,j)-th element of the adjacency matrix of $\mathcal{G}_{\sigma(t)}$, and let $B_{\sigma(t)} := \mathrm{diag}\big(b_1(\sigma(t)), b_2(\sigma(t)), \cdots, b_n(\sigma(t))\big)$ represent the leader adjacency matrix associated with $\overline{\mathcal{G}}_{\sigma(t)}$ such that $b_i(\sigma(t)) > 0$ if and only if $0 \in \mathcal{N}_i(\overline{\mathcal{G}}_{\sigma(t)})$.

In parallel with Section 2, substituting the protocols (17), (18) to the system (1), we can describe the system in the form of the Itô differential equations:

$$\begin{cases} \mathrm{d}x(t) = -h(t)(L_{\sigma(t)} + B_{\sigma(t)})x(t)\mathrm{d}t + h(t)B_{\sigma(t)}1x_0(t)\mathrm{d}t + g(t)v(t)\mathrm{d}t + h(t)G_{\sigma(t)}\mathrm{d}W_1(t) \\ \mathrm{d}v(t) = a_0(t)1\mathrm{d}t - \gamma h(t)(L_{\sigma(t)} + B_{\sigma(t)})x(t)\mathrm{d}t + \gamma h(t)B_{\sigma(t)}1x_0(t)\mathrm{d}t + \gamma h(t)G_{\sigma(t)}\mathrm{d}W_2(t) \end{cases} \quad (19)$$

where $L_{\sigma(t)}$ is the Laplacian matrix of $\mathcal{G}_{\sigma(t)}$, and

$$G_{\sigma(t)} := \mathrm{diag}\Big(\sqrt{b_1^2(\sigma(t)) + \sum_{j\in\mathcal{N}_1(\sigma(t))} \sigma_{j1}^2 a_{1j}^2(\sigma(t))}, \cdots, \sqrt{b_n^2(\sigma(t)) + \sum_{j\in\mathcal{N}_n(\sigma(t))} \sigma_{jn}^2 a_{nj}^2(\sigma(t))}\Big).$$

Let $x^* = x - x_0 1$ and $v^* = v - v_0 1$ as in Section 3.1. We get an error dynamics of (19) as follows:

$$\mathrm{d}\varepsilon(t) = F_\sigma(t)\varepsilon(t)\mathrm{d}t + G_\sigma(t)\mathrm{d}W(t), \quad t \geq 0 \quad (20)$$

where $\varepsilon(t) = (x^*(t), v^*(t))^T$, $W(t) = (W_1(t), W_2(t))^T$ and

$$F_\sigma(t) = \begin{pmatrix} -h(t)(L_{\sigma(t)} + B_{\sigma(t)}) & g(t)I_n \\ -\gamma h(t)(L_{\sigma(t)} + B_{\sigma(t)}) & 0 \end{pmatrix}, \qquad G_\sigma(t) = h(t)\begin{pmatrix} G_{\sigma(t)} \\ \gamma G_{\sigma(t)} \end{pmatrix} := h(t)\widetilde{G}_{\sigma(t)}.$$

In the sequel, we show that under switching topology, the consensus protocols (17) and (18) ensure that each follower tracks the leader in mean square. We will use the following lemma.

Lemma 2.[4] *Given $t \geq 0$ and suppose the digraph $\mathcal{G}_{\sigma(t)}$ is balanced. Then $L_{\sigma(t)} + B_{\sigma(t)} + (L_{\sigma(t)} + B_{\sigma(t)})^T$ is positive definite if and only if the vertex 0 is globally reachable in $\overline{\mathcal{G}}_{\sigma(t)}$.*

The matrix $L_{\sigma(t)} + B_{\sigma(t)} + (L_{\sigma(t)} + B_{\sigma(t)})^T$ plays a key role in the convergence analysis below. Define $\mu := \min_{t\geq 0}\big\{\lambda_{\min}(L_{\sigma(t)} + B_{\sigma(t)} + (L_{\sigma(t)} + B_{\sigma(t)})^T)\big\}$.

Prior to establishing the main result, we present a condition analogous with Assumption (A2) in Section 3.1:

(A5) There is a $\delta > 0$, such that $\frac{h(t)}{g(t)} > \frac{1}{2\gamma(1-\gamma^2)\mu} + \delta$ for $t \geq 0$.

Remark 8. *It is easily shown that $\mu > 0$ under the assumptions of Theorem 2 below by exploiting Lemma 2 and the fact that $\mathcal{H}$ is a finite set. This validates the expression in Assumption (A5).*

Theorem 2. *For system (1) with the consensus protocols (17) and (18), if for any $t \geq 0$, $\mathcal{G}_{\sigma(t)}$ is balanced, and vertex 0 is globally reachable in $\overline{\mathcal{G}}_{\sigma(t)}$, then under Assumptions (A3)-(A5), we have*

$$\lim_{t\to\infty} E\|\varepsilon(t)\|^2 = 0. \tag{21}$$

Proof. Let $\widetilde{I} := \begin{pmatrix} I_n & -\gamma I_n \\ -\gamma I_n & I_n \end{pmatrix}$. Obviously, we have $\lambda_{\min}(\widetilde{I}) = 1-\gamma$ and $\lambda_{\max}(\widetilde{I}) = 1+\gamma$. Hence $\widetilde{I}$ is a positive definite matrix by recalling $0<\gamma<1$. Define a Lyapunov function $V(t) = \varepsilon^T(t)\widetilde{I}\varepsilon(t)$.

By Itô formula and (20), we have

$$\mathrm{d}V(t) = \varepsilon^T(t)\big(\widetilde{I}F_\sigma(t) + F_\sigma^T(t)\widetilde{I}\big)\varepsilon(t)\mathrm{d}t + \mathrm{tr}(G_\sigma^T(t)\widetilde{I}G_\sigma(t))\mathrm{d}t + 2\varepsilon^T(t)\widetilde{I}G_\sigma(t)\mathrm{d}W(t).$$

Straightforward calculation yields

$$\begin{aligned} \widetilde{I}F_\sigma(t) + F_\sigma^T(t)\widetilde{I} &= -h(t)\begin{pmatrix} (1-\gamma^2)\big(L_{\sigma(t)} + B_{\sigma(t)} + (L_{\sigma(t)} + B_{\sigma(t)})^T\big) & -I_n\frac{g(t)}{h(t)} \\ -I_n\frac{g(t)}{h(t)} & 2\gamma I_n\frac{g(t)}{h(t)} \end{pmatrix} \\ &:= -h(t)\widetilde{Q}_\sigma(t). \end{aligned}$$

By using the Haynsworth inertia additivity formula [21], Assumption (A5), we get that $\widetilde{Q}_\sigma(t)$ is positive definite with $\nu := \min_{t\geq 0}\{\lambda_{\min}(\widetilde{Q}_\sigma(t))\} > 0$.

Therefore, we have

$$\mathrm{d}V(t) \leq -h(t)\frac{\nu}{1+\gamma}V(t)\mathrm{d}t + h^2(t)\mathrm{tr}\big(\widetilde{G}_{\sigma(t)}^T\widetilde{I}\widetilde{G}_{\sigma(t)}\big)\mathrm{d}t + 2h(t)\varepsilon^T(t)\widetilde{I}\widetilde{G}_{\sigma(t)}\mathrm{d}W(t). \tag{22}$$

The remaining proofs are similar with those in Theorem 1 by noting that $\mathcal{H}$ is a finite set, and hence omitted. □

Remark 9. *From Theorem 2 we see that the designed protocol may guarantee the state of each follower tracks that of the leader in mean square even under the switching topology.*

4. Numerical example

In this section, we provide a numerical simulation to illustrate the theoretical results.

We consider a network consisting of four agents including one leader labeled by vertex 0, as shown in Fig. 1. The digraph in this figure is assumed to have $0-1$ weights. With simple calculation, it is not hard to solve P from Equation (13) and obtain $\lambda_{\max}(P) = 0.9447$. We take $\sigma_{ij} = 0.1$ for all i, j, $h(t) = \frac{1}{t+2}$, $g(t) = \frac{1}{6(t+2)}$ and $\gamma = 0.5$. Therefore, Assumptions (A1)-(A4) in Theorem 1 hold.

The simulation results for the consensus errors x^* and v^* are shown in Fig. 2 and Fig. 3 respectively, with initial value $\varepsilon(0) = (-2, 1.5, 3, 2, -1.5, -1)^T$. From Fig. 2 and Fig. 3, we can see that three followers can eventually follow the leader.

5. Conclusion

This paper studies a leader-following coordination problem for multi-agent systems with a time-varying leader under measurement noises. Although the state of the leader keeps changing and the measured information by each follower is corrupted by white noises, we propose a neighborhood-based protocol for each agent to follow the leader. We present sufficient conditions for each follower to track the leader in mean square under directed fixed topologies. Sufficient conditions are also provided when the interaction topology

is switching and the subgraph formed by the followers is balanced. Finally, numerical simulations are presented to illustrate the theoretical results. Topics worth investigating in the future include time-delay cases and the design of almost sure consensus protocols.

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Figure captions

Fig. 1 Directed network $\overline{\mathcal{G}}$ of four agents involving one leader. $\overline{\mathcal{G}}$ has $0-1$ weights.
Fig. 2 Consensus error x^* for the agents.
Fig. 3 Consensus error v^* for the agents.